PSEUDO-VALUATION DOMAINS : A SURVEY

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1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. We begin by recalling some background material. As in [31], an integral domain R, with quotient field K, is called a *pseudo-valuation domain (PVD)* in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. Pseudo-valuation domains have been studied extensively in [32], [24], [3], [4], [6], [30], [28], [7] [1], [2], [35], [36], [33], [34], [9], [10], [12], [13], [38], and [19]. In [8], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [8] and [5] that a prime ideal P of R is said to be *strongly prime (in R)* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring (PVR)* if each prime ideal of R is strongly prime. A PVR is necessarily quasilocal [8, Lemma 1(b)]; a chained ring(recall that a ring R is said to be a chained ring if for every $a, b \in R$, either $a \mid b$ or $b \mid a$) is a PVR [[8], Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [3, Proposition 3.1], [4, Proposition 4.2], and [10, Proposition 3]). Recall from [11] and [25] that a prime ideal P of R is called *divided* if it is comparable (under inclusion) to every ideal of R. A ring R is called a *divided ring* if every prime ideal of R is divided. In [14], the author gave another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors).

Recall from [14] that for a ring R with total quotient ring $T(R)$ such that $Nil(R)$ is a divided prime ideal of R, let $\phi : T(R) \longrightarrow K := R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal Q of $\phi(R)$ is called a *K-strongly prime* if $xy \in Q$, $x \in K, y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$ is called a *K-pseudo-valuation ring (K-PVR)*. A prime ideal P of R is called a *ϕ-strongly prime* if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring ($\phi - PVR$)*. It is shown in [14, Corollary 7(2)] that a ring R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal and for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid ac$ in R for each nonunit $c \in R$. Also, it is shown in [15, Theorem 2.6] that for each $n \geq 0$ there is a ϕ-PVR of Krull dimension n that is not a PVR. ϕ-pseudo-valuation rings have been studied extensively in [14], [15], [16], [17], and [22]. We would like to point out that if R is an integral domain, then Dobbs, Fontana, Huckaba, and Papick in [30] have defined and studied "T-strongly primes" and "strong rings" (see section 4) . Chang [22] gave another generalization of pseudo-valuation domains. Recall from [22] that a Marot ring R with total quotient ring T(R) is called an r-pseudo-valuation ring (r-PVR) if each regular prime ideal P of R is r-strongly prime, in the sense that $xy \in P$, $x \in T(R)$, $y \in T(R)$ implies that either $x \in P$ or $y \in P$. Chang [22] gave an example of an r-PVR that is not a ϕ-PVR.

In this article, we will only study and survey pseudo-valuation domains. If the reader is interested in the generalization of pseudo-valuation domains to the context of an arbitrary rings with nonzero zerodivosors, then we recommend the following papers : [8], [5], [11], [12], [13], [14], [15], [16], [17], [18], and [22].

We begin by stating some simple properties and characterizations of pseudo-valuation domains (PVDs). Recall that an integral domain R is called a valuation domain if for every $a, b \in R$, either $a \mid b$ or $b \mid a$.

PROPOSITION 2.1. *(*[31, Proposition 1.1]. *Every valuation domain is a pseudo-valuation domain.* □

The following proposition is a characterization of strongly prime ideals.

PROPOSITION 2.2. *(*[31, Proposition 1.2]*). Let P be a prime ideal of a domain R with quotient field K. Then P is strongly prime if and only if $x^{-1}P \subset P$ whenever $x \in K \setminus R$.* □

PROPOSITION 2.3. *(*[31, Corollary 1.3]*). In a pseudo-valuation domain R, the prime ideals are linearly ordered (under inclusion). In particular, R is quasilocal.* □

Anderson [4, Proposition 4.6] gave the following characterization of nonprincipal strongly prime ideals:

PROPOSITION 2.4. *(*[4, Proposition 4.6]*). Let R be an integral domain with quotient field K, and let I be a nonzero ideal of R. Then the following statements are equivalent:*

1. *I is a nonprincipal strongly prime ideal.*
2. *$I^{-1} = \{x \in K : xI \subset R\}$ is a ring and I is comparable to each principal fractional ideal of R.* □

Let I be an ideal of an integral domain R with quotient field K. Then $I : I = \{x \in K : xI \subset I\}$. Another characterization of strongly prime ideals was given in [1, Proposition 1.3]:

PROPOSITION 2.5. *(*[1, Proposition 1.3], *also see* [3, Proposition 4.2 and 4.3]*) . Let P be a prime ideal in an integral domain R with quotient field K. Then the following statements are equivalent :*

1. *P is a strongly prime ideal.*
2. *$S = K \setminus P$ is multiplicatively closed.*
3. *P is prime and is comparable to each fractional ideal of R.*
4. *$P : P$ is a valuation domain with maximal ideal P.*
5. *P is a prime ideal in some valuation overring of R.* □

PROPOSITION 2.6. *(Proposition 2.5 and* [35, Theorem 7] *and* [5, Corollary 3.7(b)]*). Let P be a strongly prime ideal of an integral domain R. Then $R_P = P : P$ if and only if R_P is a valuation domain. In particular, if P is a nonmaximal ideal of R, then $R_P = P : P$ is a valuation domain.* □

PROPOSITION 2.7. *(*[35, Theorem 1]*). Let R be a pseudo-valuation domain, and let I be an ideal of R and P be a prime ideal of R such that $P \subset I$. Then P is a prime ideal of $I : I = \{x \in K : xI \subset I\}$.* □

PROPOSITION 2.8. *(*[31, Theorem 1.4 and Theorem 1.5]*). Let R be an integral domain with quotient field K. The following statements are equivalent:*

1. *R is a pseudo-valuation domain.*
2. *A maximal ideal of R is strongly prime.*
3. *For each $x \in K \setminus R$ and for each nonunit a of R, we have $x^{-1}a \in R$.* □.

The following proposition is a restatement of Proposition 2.8(3).

PROPOSITION 2.9. *(*[10, Proposition 3(4)]*) An integral domain is a PVD if and only if for every $a, b \in R$, either $a \mid b$ or $b \mid ac$ for every nonunit c of R.* □.

PROPOSITION 2.10. *([31, Corollary 2.9], also see [9, Proposition 5]). If a pseudo-valuation domain R has a nonzero principal prime ideal, then R is a valuation domain.* □.

Hedstrom and Houston [31, Theorem 2.10] gave the following characterization of pseudo-valuation domains:

PROPOSITION 2.11. *([31, Theorem 2.10]). Let (R, M) be a quasilocal domain with quotient field K which is not a valuation domain. Then R is a pseudo-valuation domain if and only if $M^{-1} = \{x \in K : xM \subset R\}$ is a valuation domain with maximal ideal M.* □

Anderson and Dobbs [6, Proposition 2.5] sharpened the above Proposition.

PROPOSITION 2.12. *([6, Proposition 2.5]). Let (R, M) be a quasilocal domain with quotient field K. Then R is a pseudo-valuation domain if and only if $M : M = \{x \in K : xM \subset M\}$ is a valuation domain with maximal ideal M.* □

Anderson [4, Proposition 4.1] gave this characterization of pseudo-valuation domains:

PROPOSITION 2.13. *([4, Proposition 4.1]). Let R be an integral domain with quotient field K. The following statements are equivalent:*

1. *R is a pseudo-valuation domain, (and hence quasilocal).*
2. *For each $x \in K$ and prime ideal P of A, xA and P are comparable (under inclusion).* □

If R is a ring, then $U(R)$ denotes the set of all units of R. Anderson and Anderson [1, Theorem 1.2] gave the following characterization of pseudo-valuation domains:

PROPOSITION 2.14. *([1, Theorem 1.2]). Let K be a field and R be a subring of K with group of units $U(R)$. Then $S = (K \setminus R) \cup U(R)$ is multiplicatively closed if and only if either R is a pseudo-valuation domain with quotient field K or R is a subfield of K.* □

Let b be an element of an integral domain R. Then an element d of R is called a proper divisor of b if $b = dm$ for some nonunit $m \in R$. Badawi[12, Proposition 4] gave the following characterization of pseudo-valuation domains:

PROPOSITION 2.15. *([12, Proposition 4]). An integral domain R is a pseudo-valuation domain if and only if for every $a, b \in R$, either $a \mid b$ or $d \mid a$ for every proper divisor d of b.* □

Anderson and Dobbs [6, Proposition 2.6] showed that a pseudo-valuation domain is the pullback of a valuation domain:

PROPOSITION 2.16. *([6, Proposition 2.6]). Let V be a valuation domain with maximal ideal M, $F = V/M$ its residue field, $\phi : V \longrightarrow F$ the canonical epimorphism, k a subfield of F, and $R = \phi^{-1}(k)$. Then the pullback $R = V \times_F k$ is a pseudo-valuation domain.* □

In the following proposition, Dobbs [24, Proposition 4.9] gave an extension of Hedstrom-Houston's observation [31, Example 2.1] that the $D + M$-construction yields a pseudo-valuation domain whenever D is a field.

PROPOSITION 2.17. *([24, Proposition 4.9]). Let $M \neq 0$ be the maximal ideal of a valuation domain $V = K + M$, where K is a field. Let D be a proper subring of K. Set $R = D + M$. Then R is a pseudo-valuation domain if and only if either D is a pseudo-valuation domain with quotient field K or D is a field.*

2.1. **Examples of pseudo-valuation domains.** Hedstrom and Houston gave the following example of a pseudo-valuation domain that is not a valuation domain:

EXAMPLE 2.1.1. *([31, example 2.1]). Let V be a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V. If F is a proper subfield of K, then $R = F + M$ is a pseudo-valuation domain that is not a valuation domain. In particular, if K is a field and F is a proper subfield of K, then $F + XK[[X]]$ is a pseudo-valuation domain that is not a valuation domain. For example, $Q + XR[[X]]$ is a pseudo-valuation domain that is not a valuation domain.* □

EXAMPLE 2.1.2. *([32, Example 3.1]). For each positive integer n (possibly infinite), there is a pseudo-valuation domain of Krull dimension n that is not a valuation domain. Let $D = Q + XR[[X]]$. Then D is a pseudo-valuation domain of Krull dimension 1 that is not a valuation domain. Now, assume that $n > 1$. Let K be the quotient field of D. Then there is a valuation domain of the form $K + M$ with maximal ideal M of Krull dimension $n - 1$. Then $R = D + M$ is a pseudo-valuation domain by Proposition 2.17. By standard properties of the $D + M$-construction, R is not a valuation domain and R has n Krull dimension.*

3. OVERRINGS THAT ARE PSEUDO-VALUATION DOMAINS

Recall that if R is an integral domain with quotient field K, then we say that B is an overring of R if $R \subset B \subset K$. We start with the following proposition:

4.2 **PROPOSITION 3.1.** *([31, Proposition 2.6]. Let R be a pseudo-valuation domain with maximal ideal M. If P is a nonmaximal prime ideal of R, then R_P is a valuation domain,(and hence a pseudo-valuation domain).* □.

PROPOSITION 3.2. *([3, Proposition 4.3], also see [9, Proposition 6]). Let R be an integral domain with quotient field K. Supopose that P is a nonzero strongly prime ideal of R. Then :*

1. *If P is not principal, then $P^{-1} = \{x \in K : xP \subset R\} = P : P = \{x \in K : xP \subset P\}$ is a valuation domain.*
2. *If P is principal, then $P : P = R$ is a valuation domain.* □.

Anderson, Badawi, and Dobbs [8, Lemma 20] showed the following:

(1) **PROPOSITION 3.3.** *([8, Lemma 20]. Let R be a pseudo-valuation domain with maximal ideal M. Let B be an overring of R. If $s^{-1} \in B$ for some nonzero $s \in M$, then B is a pseudo-valuation domain.* □

Let R' be the integral closure of R.

PROPOSITION 3.4. *([31, Proposition 2.7], [24, Proposition 4.2]). Let R be a pseudo-valuation domain with maximal ideal M. Then $R' = M : M$ if and only if every overring of R is a pseudo-valuation domain.* □

Badawi showed the following:

PROPOSITION 3.5. *([12, Corollary 18]). Let R be a pseudo-valuation domain with maximal ideal M. Then the following statements are equivalent:*

1. *$R' = M : M$.*
2. *Every overring of R is a pseudo-valuation domain.*
3. *Every overring C of R such that $C \subset M : M$ is a pseudo-valuation domain.*
4. *Every overring C of R such that $C \subset M : M$ is a pseudo-valuation domain with maximal ideal M.*
5. *M is the maximal ideal of every overring C of R such that $C \subset M : M$.*

6. *$R \subset C$ satisfies the INC condition for every overring C of R such that $C \subset M : M$. (Recall that $R \subset C$ satisfies the INC condition if any two prime ideals of C with the same contraction in R are incomparable (under inclusion).)*

Anderson, Badawi, and Dobbs showed the following:

PROPOSITION 3.6. *([5, Corollary 2.2] If (R, M) is a pseudo-valuation domain, then the following conditions are equivalent :*

1. *$R' = M : M$.*
2. *Every overring of R is a pseudo-valuation domain.*
3. *Every overring of R that does not contain an element of the form $1/s$ for some $s \in M$ is a pseudo-valuation domain.*
4. *For each $u \in (M : M) \setminus R$, $R[u]$ is a pseudo-valuation domain.*
5. *For each $u \in (M : M) \setminus R$, $R[u]$ is quasilocal.*
6. *Every overring of R is quasilocal.* □

Badawi[13, Theorem 3] proved the following result:

PROPOSITION 3.7. *([13, Theorem 3]). Let (R, M) be a pseudo-valuation domain with quotient field K, and let V be a valuation domain with maximal ideal N such that $R \subset V \subset K$. If $P = N \cap R$ is different from M, then $V = R_P$.* □

The above result was used to prove the following:

PROPOSITION 3.8. *([13, Theorem 8]). Let (R, M) be a pseudo-valuation domain with quotient field K. The following are equivalent:*

1. *$R' = M : M$.*
2. *Every valuation domain V of R other than $M : M$ such that $R \subset V \subset K$ is of the form R_P for some nonmaximal prime ideal P of R.*

3. *Every overring of R is a pseudo-valuation domain.*

Recall that an overring B of an integral domain R is called a proper overring of R if $R \neq B$. Let R be an integral domain with quotient field K. Okabe[36] defined R to be a quasi-valuation domain (QVD) if each proper quasilocal overring B of R with maximal ideal M_B satisfies the condition (QV) $R : B = \{x \in K : xB \subset R\} = M_B$. Okabe showed the following:

PROPOSITION 3.9. *([36, Proposition 1]). Every Valuation domain is a quasi-valuation domain.* □

Using the concept of quasi-valuation domains, Okabe proved the following:

PROPOSITION 3.10. *([36, Theorem 8]). Let R be a quasilocal domain with maximal ideal M and quotient field K. Then the following conditions are equivalent:*

1. *R is a quasi-valuation domain.*
2. *Each overring of R is a pseudo-valuation domain.*
3. *Each proper valuation overring V of R satisfies* (QV).
4. *Each proper minimal valuation overring of R satisfies* (QV).
5. *some proper minimal valuation overring of R satisfies* (QV).
6. *$R' = M^{-1} = \{x \in K : xM \subset R\}$.*

Recall that an integral domain R with quotient field K is called seminormal if whenever $x^2, x^3 \in R$ for some $x \in K$, then $x \in R$. Anderson, Dobbs, and Huckaba proved the following result:

PROPOSITION 3.11. *([7, Proposition 3.1]).*

1. *Each pseudo-valuation domain is seminormal.*
2. *Let R be a pseudo-valuation domain with maximal ideal M and quotient field K. Then the **following** four conditions are equivalent:*

(a) *For each $\alpha \in K \setminus R$, each overring of R which is maximal without α is a pseudo-valuation domain.*

(b) *Each overring of R is seminormal .*

(c) *Each overring of R is a pseudo-valuation domain.*

(d) $R' = M : M$. □

Let R be an integral domain with quotient field K. Dobbs and Fontana[28] defined R to be a locally pseudo-valuation domain (LPVD)if R_P is a pseudo-valuation domain for every (nonzero) prime ideal P of R. For a generalization of locally pseudo-valuation domains to the context of arbitrary rings with nonzero zerodivisors see [18]. Dobbs and Fontana showed the following:

PROPOSITION 3.12. *([28, Proposition 2.2]). An integral domain R is a locally pseudo-valuation domain if and only if R_M is a pseudo-valuation domain for every maximal ideal M of R.* □

PROPOSITION 3.13. *([28, Example 2.5]). Let $n \geq 2$. Then there exists a locally pseudo-valuation domain R with precisely n maximal ideals, such that R is neither a pseudo-valuation domain nor a Prüfer domain.*

Proof. (sketch). Let k be a field with the following two properties: (1) there exist n pairwise incomparable valuation domains $V_i = k + M_i$ having (maximal ideal M_i, residue class field k and) a common quotient field; (2) there exists n distinct proper subfields k_i of k. Then $R = \cap(k_i + M_i)$ is a locally pseudo-valuation domain, and R is neither a Prüfer domain nor a pseudo-valuation domain. □

Let R be an integral domain with quotient field K. Recall from [37] that R is said to be an *i-domain* if the contraction map $i : Spec(S) \longrightarrow Spec(R)$ is an

injection for each overring S of R; equivalently ([37, Corollary 2.15]), if the integral closure of R_M is a valuation domain for each maximal ideal M of R.

PROPOSITION 3.14. *([28, Theorem 2.9]). Let R be an integral domain. Then the following conditions are equivalent:*

1. *Each overring of R is a locally pseudo-valuation domain.*
2. *R is a locally pseudo-valuation domain and each overring of R is seminormal.*
3. *R is a locally pseudo-valuation domain and R' is a Prüfer domain.*
4. *R is a locally pseudo-valuation domain and an i-domain.* □

PROPOSITION 3.15. *([28, Corollary 2.10]). Let R be a pseudo-valuation domain with maximal ideal M. Then the following conditions are equivalent:*

1. *Each overring of R is seminormal.*
2. *Each overring of R is a locally pseudo-valuation domain.*
3. *Each overring of R is a pseudo-valuation domain.*
4. $R' = M : M$. □.

Let R be an integral domain with quotient field K. Matsuda[33] called an overring of R which is maximal without a specified element of $K \setminus R$ a specified overring *s-overring*. Matsuda[33] showed the following:

PROPOSITION 3.16. *([33, Theorem 3]). Assume that R is a domain with R' is quasilocal. Then if each s-overring of R is a pseudo-valuation domain, then each overring of R is a pseudo-valuation domain.* □

PROPOSITION 3.17. *([33, Theorem 4]). Let R be an integral domain with quotient field K. Then the following conditions are equivalent:*

1. *Each s-overring of R is a pseudo-valuation domain.*

2. *Each s-overring B of R is a pseudo-valuation domain with maximal ideal M_B, and $B' = M_B : M_B$.*
3. *Each s-overring B of R is an i-domain, and each integral overring of B is seminormal.*
4. *For each s-overring B of R, each integral overring of B is seminormal, and B' is a pseudo-valuation domain.*
5. *Each overring of R is seminormal, and, for each overring B of R which is not an i-domain, B' contains no s-overring of B.*
6. *Each overring of R is seminormal, and each s-overring of R is an i-domain.*
7. *Each integral overring of R is seminormal, R' is a Prüfer domain, and each s-overring of R is an i-domain.*
8. *For each maximal ideal M of R, each s-overring of R_M is a pseudo-valuation domain.*

Let R be an integral domain. Recall that R is called *t-closed* if, whenever $a, r, c \in R$ satisfies $a^3 + arc - c^2 = 0$, there exists $b \in R$ such that $b^2 - rb = a$, and $b^3 - rb^2 = c$. Picavet[38] showed the following:

PROPOSITION 3.18. *(*[38, Proposition 3.1]*). Let R be a pseudo-valuation domain with maximal ideal M and quotient field K. Then:*

1. *R is t-closed.*
2. *The following conditions are equivalent:*
   (a) *Each overring of R is a pseudo-valuation domain.*
   (b) *Each overring of R is t-closed.*
   (c) *Each overring of R is seminormal.*
   (d) *$R' = M : M$.*

   (e) *For each $\alpha \in K \setminus R$, each overring of R which is maximal without α is a pseudo-valuation domain.* □

4. Atomic pseudo-valuation domains

Let R be an integral domain. Recall that a nonunit a of R is called an *atom* of R if a is an *irreducible* element of R. If each nonunit element of R is a product of atoms of R, then R is called an *atomic domain*. It is well-known that a Noetherian domain is an atomic domain. Hedstrom and Houston[31] showed the following:

PROPOSITION 4.1. *(*[31, Theorem 3.1]*). Let R be a Noetherian domain with quotient field K and integral closure R'. Then R is a pseudo-valuation domain if and only if R' is a valuation domain.* □

PROPOSITION 4.2. *(*[31, Proposition 3.2]*). If R is a Noetherian pseudo-valuation domain which is not a field, then R has Krulll dimension 1.* □

PROPOSITION 4.3. *(*[31, Corollary 3.3]*). If R is a Noetherian pseudo-valuation domain, then every overring of R is a pseudo-valuation domain.* □

Let R be an atomic integral domain. Anderson and Mott [2] called a subset S of R a *universal* if each element of S is divisible by each atom of R. Anderson and Mott in [2] showed the following:

PROPOSITION 4.4. *(*[2, Theorem 5.1]*). Let R be an atomic quasilocal domain with maximal ideal M. Then R is a pseudo-valuation domain if and only if M^2 is universal.* □

The following result is a stronger version of Proposition 4.2:

PROPOSITION 4.5. *([2, Corollary 5.2] and [12, Theorem 9]) and [23]). If R is an atomic pseudo-valuation domain which is not a field, then R has Krull dimension 1.* □

Recall from [39] that an atomic integral domain R is called a half-factorial domain (HFD) if each factorization of a nonzero nonunit element of R into a product of irreducible elements (atoms) of R has the same length. Let R be a half-factorial domain and x be a nonzero element of R. Then we define $L(x) = n$ if $x = x_1x_2...x_n$ for some atoms x_i of R. If x is a unit of R, then $L(x) = 0$. We have the following:

PROPOSITION 4.6. *([2, Theorem 6.2], also see [12, Theorem 5]). If R is an atomic pseudo-valuation domain, then R is a half-factorial domain.*

Badawi [12] gave a characterization of atomic pseudo-valuation domains in terms of half-factorial domains.

PROPOSITION 4.7. *([12, Theorem 6]). Let R be an atomic domain. Then the following statements are equivalent:*

1. *R is a pseudo-valuation domain.*
2. *R is a half-factorial domain and for every $x, y \in R$, if $L(x) < L(y)$, then $x \mid y$ in R.* □

4.1. Examples of atomic pseudo-valuation domains.

EXAMPLE 4.1.1. *([31, Example 3.6]). Let $R = Z[\sqrt{(5)}]_{(2, 1+\sqrt{(5)})}$. Then R is a Noetherian (and hence atomic) pseudo-valuation domain.*

EXAMPLE 4.1.2. *([2]). Let k be any field and X, Y be indeterminates. Then $R = k + Xk(y)[[X]]$ is an atomic pseudo-valuation domain that is not Noetherian.*

For further study on examples of pseudo-valuation domains, we recommend [29] and [23].

5. Related results

Let R be a subring of an integral domain T. Dobbs, Fontana, Huckaba, and Papick [30] called a prime ideal P of R *T-strong* if, whenever $x \in T$ and $y \in T$ satisfy $xy \in P$, then either $x \in P$ or $y \in P$. If each prime ideal of R is T-strong, then T is called a *strong extension of* R (or $R \subset T$ is a strong extension). Evidently, an integral domain R with quotient field K is a pseudo-valuation domain if and only if $R \subset K$ is a strong extension. The following is an example of of a strong overring extension $R \subset T$ of domains for which neither R nor T is quasilocal, (and hence neither R nor T is a pseudo-valuation domain.

EXAMPLE 5.1. *([30, Example 2.1]). Let L be the quotient field of $Z[X]$, and $V = L + XL[[X]]$ (observe that V is a valuation domain with maximal ideal $XL[[X]]$). Set $R = Z + XL[[X]]$ and $T = Z[X] + XL[[X]]$. Then $R \subset T$ is a strong overring extension with the stated properties. $T = Z[X] + XL[[X]]$.*

Recall from [25] that a prime ideal P of an integral domain R is said to be *divided* in R if P is comparable (under inclusion) with each principal ideal of R. The following result is stated in [30]:

PROPOSITION 5.2. *([30, Theorem 2.3]). Let P be a prime ideal of an integral domain R. Then $R \subset R_P$ is a strong extension if and only if both P is divided in R and R/P is a pseudo-valuation domain. Furthermore, if $R \subset R_P$ is a strong extension, then the set of all prime ideals of R which contain P is linearly ordered by inclusion and R is quasilocal.* □

The following result [30] states a characterization of pseudo-valuation domains in terms of strong extensions:

PROPOSITION 5.3. *([30, Theorem 2.9]). A domain R is a pseudo-valuation domain if and only if R has a prime ideal P satisfying the following two conditions:*

1. *$R \subset R_P$ is a strong extension; and*
2. *R_P is a valuation domain. Recall that if A is a ring then $U(A)$ denotes the set of all units of A.* □

PROPOSITION 5.4. *([30, Theorem 3.1]). Let R be an integral domain which is distinct from its quotient field K; and T is an integral domain contains R. If $K \subset T$, then $R \subset T$ is a strong extension if and only if both R is a pseudo-valuation domain and $U(T) = U(K)$.* □

Let R be an integral domain. Recall that an ideal I of R is called a cancellation ideal if, whenever $IJ_1 = IJ_2$, then $J_1 = J_2$. Also, recall that an ideal I of R is called a quasi-cancellation ideal, if $aI \subset IJ$ for some $a \in R$ and a finitely generated ideal J of R, then $a \in J$. Matsuda and Sugatani [34] proved the following:

PROPOSITION 5.5. *([34, Summary]).*

1. *For a pseudo-valuation domain R, a nonzero ideal I of R is a cancellation ideal if and only if I is a principal ideal.*
2. *There is a pseudo-valuation domain R that is not a valuation domain, such that R has a quasi-cancellation ideal which is not a cancellation ideal.* □

Badawi and Houston in [19] called an ideal I of an integral domain R with quotient field K *powerful* if, whenever $x \in K$, $y \in K$, and $xy \in I$, then either $x \in I$ or $y \in I$.

PROPOSITION 5.6. *([19, Proposition 1.3, and Corollary 1.6]). A prime ideal of R is strongly prime if and only if it is powerful. In particular, an integral domain R is a pseudo-valuation domain if and only if a maximal ideal of R is powerful.* □

PROPOSITION 5.7. *([19, Proposition 1.14]). Let I be a powerful ideal of an integral domain R, and suppose that $P \subset I$ is a nonzero finitely generated prime ideal of R. Then R is a pseudo-valuation domain.* □

Recall that R' denotes the integral closure of an integral domain R inside its quotient field.

PROPOSITION 5.8. *([19, Theorem 1.15]). Suppose that an integral domain R with quotient field K admits a powerful ideal I and that $M = Rad(I) = \{x \in R : x^n \in I \text{ for some } n \geq 1\}$ is a maximal ideal of R. Then :*

1. *R is quasilocal with maximal ideal M.*
2. *$IR' \subset M$, and therefore IR' is an ideal of R.*
3. *R' is a pseudo-valuation domain with maximal ideal $N = Rad(IR')$, and hence $N : N = \{x \in K : xN \subset N\}$ is a valuation overring of R with maximal ideal N.* □

Recall from [19] that an ideal I of an integral domain R with quotient field K is called *strongly primary* if, whenever $xy \in I$ for some $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$. An integral domain R is called *almost pseudo-valuation domain (APVD)* if every prime ideal of R is strongly primary. Also, recall from [25] that a prime ideal of R is called divided if it is comparable to every principal ideal of R. If every prime ideal of R is divided, then R is called a divided domain. Dobbs in [24] proved that a pseudo-valuation domain is a divided domain. Badawi and Houston in [19] showed the following.

PROPOSITION 5.9. ([19, Proposition 3.2]). *Let R be an almost pseudo-valuation domain. Then R is a (quasilocal) divided domain. Moreover, every nonmaximal prime ideal of R is strongly prime.* □

PROPOSITION 5.10. ([19, Theorem 3.4]). *The following statements are equivalent for an integral domain R:*

1. *R is almost pseudo-domain.*
2. *Some maximal ideal of R is strongly primary.*
3. *R is a quasilocal domain, and the maximal ideal M of R is such that $M : M$ is a valuation domain with M is primary to the maximal ideal of $M : M$.* □

PROPOSITION 5.11. ([19, Proposition 3.7]). *If R is an almost pseudo-valuation domain with maximal ideal M, then R' is a pseudo-valuation domain with maximal ideal M.* □

PROPOSITION 5.12. ([19, Proposition 3.8]). *If each overring of an integral domain R is an almost pseudo-valuation domain, then R' is a valuation domain.* □

The converse of the Proposition 5.12 is false (see [19, Example 3.9].) However, we state the following result:

PROPOSITION 5.13. ([19, Proposition 3.10]). *Let R be an almost pseudo-valuation domain, and assume that every integral overring of R is an almost pseudo-valuation domain. Then every overring of R is an almost pseudo-valuation domain.* □

In [21] Bastida and Gilmer prove that a domain R shares an ideal with a valuation overring iff each overring of R which is different from the quotient field K of R has a nonzero conductor to R. Domains with this property, called *conducive domains*, were explicitly defined and studied by Dobbs and Fedder [26] and further studied by Barucci, Dobbs, and Fontana [20] and [27]. Recall from [26] that an integral domain R with quotient field K is called a *conducive domain* if for each overring T of R, the conductor $R : T = \{x \in K : xT \subset R\}$ is nonzero. Badawi and Houston in [19] showed that conducive domains, powerful ideals, and strongly primary ideals are intimately connected.

PROPOSITION 5.14. ([19, Theorem 4.1]). *The following statements are equivalent:*

1. *R is a conducive domain.*
2. *R admits a powerful ideal.*
3. *R admits a strongly primary ideal.*
4. *R shares a nonzero ideal with some conducive overring.* □

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